Connes' Gauge Theory on Noncommutative Space-Times

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Abstract

Connes' gauge theory is defined on noncommutative space-times. It is applied to formulate a noncommutative Glashow-Weinberg-Salam (GWS) model in the leptonic sector. It is shown that the model has two Higgs doublets and the gauge bosons sector after the Higgs mechanism contains the massive charged gauge fields, two massless and two massive neutral gauge fields. It is also shown that, in the tree level, the neutrino couples to one of two 'photons', the electron interacts with both 'photons' and there occurs a nontrivial ν_R-interaction on noncommutative space-times. Our noncommutative GWS model is reduced to the GWS theory in the commutative limit. Thus in the neutral gauge bosons sector there are only one massless photon and only one Z^0 in the commutative limit.

§1. Introduction

Connes' reconstruction [1), 2), 3), 4), 5)] of the standard model assumes [1)] the two-sheeted Minkowski space-time $M_4 \times Z_2$, the two sheets being separated by the inverse of order of the weak scale, while the Minkowski space-time M_4 is assumed to be continuous. On the other hand, there is a growing attention to a possibility [6), 7), 8), 9), 10)] that our present space-time geometry would change and the space-time coordinates become noncommutative at very short distances. The non-commutativity scale is fundamentally different from the weak scale and supposed [7)] to be of order of the Planck length. The noncommutative geometry [2)] provides us with a suitable mathematical framework to describe such a noncommutative space-time structure. In this paper we ask ourselves how the two different scales appear in the noncommutative gauge theories (NCGT) [9), 10), 11), 12), 13), 14), 15), 16)] by extending Connes' gauge theory on $M_4 \times Z_2$ in the framework of NCGT.

On noncommutative space-times characterized by the commutation relations for the hermitian coordinate operators $\hat{x}^\mu$

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, 3, \tag{1·1}$$

where $\theta^{\mu\nu}$ is a real antisymmetric tensor commuting with $\hat{x}^\rho$, the spinor $\psi(x)$ should be regarded as an operator-valued function $\psi(\hat{x})$, which is an element of an algebra $\mathcal{A}_x$ of functions in $\hat{x}^\mu$ modulo the relations (1·1), and the partial derivative $\partial_\mu \psi(\hat{x})$ is to be replaced [17)] by the commutator $[\hat{p}_\mu, \psi(\hat{x})]$, where $\hat{p}_\mu$ is defined by

$$\hat{p}_\mu = -i\theta_{\mu\nu}\hat{x}^\nu, \qquad \theta_{\mu\nu}\theta^{\nu\lambda} = \delta_\mu^{\ \lambda}, \qquad [\hat{p}_\mu, \hat{x}^\nu] = \delta_\mu^{\ \nu}. \tag{1·2}$$

Here and hereafter we assume that the matrix $\theta = (\theta^{\mu\nu})$ is invertible.

There arise new features in NCGT apart from its nonlocality. The most prominent one is that the noncommutative $U(1)$ has a field strength of Yang-Mills (YM) type. [9), 10), 11), 13), 14), 15), 16)] The other is that the YM action but not the YM Lagrangian are gauge-invariant. Similarly, if the gauge transformation for $\psi(\hat{x})$ is acted upon also from the right, namely, $\psi(\hat{x}) \to g(\hat{x})\psi(\hat{x})u^\dagger(\hat{x})$ provided that the matrix multiplication is consistently calculable, only the Dirac action becomes gauge-invariant. We shall argue that, if the fermion mass is not gauge-invariant, the combination of the left and right actions determines the pattern of the Higgs mechanism generating the input fermion mass, yielding a different scale from that determining the commutation relations (1·1).

Connes' interpretation [3)] of the standard model regards the Hilbert space of spinors and their charge conjugates as a module over the algebra $\mathcal{A} \otimes \mathcal{A}^o$, $\mathcal{A}^o$ being the opposite algebra of the color-flavor algebra $\mathcal{A}$. This essentially means a factorization of the gauge transformation for the

doubled spinor [18] in such a way that each factor contains flavor and color, separately, while an Abelian factor is present in both. The unitary group of the algebra $\mathcal{A}$ has two $U(1)$s, whereas the standard model gauge group possesses only one. This leads to one additional requirement, the unimodularity condition [2), 3)], to reconstruct the standard model in Connes' scheme. As we have shown recently [19], it happens to determine the correct hypercharge assignment uniquely if ν_R exist in each generation. In this paper, considering the leptonic sector only, we shall show that the factorization is naturally obtained by the *two-sided* gauge transformation without introducing the doubled spinor.

In the next section we define Connes' YM on noncommutative space-times in the operator formalism and apply it to formulate a noncommutative Glashow-Weinberg-Salam (GWS) model in the leptonic sector, which contains two Higgs doublets. In order to study the Higgs mechanism in our noncommutative GWS model, we rewrite the noncommutative Connes' YM in terms of the Weyl-Moyal description [20), 21)] in §4. It turns out that the model contains two massless and two massive neutral gauge fields in addition to the charged ones in the gauge bosons sector. The neutral components become a single massless and a single massive neutral gauge fields in the commutative limit [*]. Similarly the two Higgs doublets become related, leaving a single standard Higgs doublet, in the commutative limit. The final section is devoted to discussions. There are two technical Appendices.

§2. Noncommutative Dirac-Yukawa action and noncommutative Connes' YM

The free Dirac action reads

$$\hat{S}_D = (2\pi)^2\sqrt{\det\theta}\,\mathrm{tr}\bar{\psi}(\hat{x})(i\gamma^\mu[\hat{p}_\mu,\psi(\hat{x})] - M\psi(\hat{x})) = \int d^4x\bar{\psi}(x)D\psi(x), \tag{2·1}$$

where $D = D_0 - M$, $D_0 = i\partial\!\!\!/\otimes 1_n$, 1_n being the n-dimensional unit matrix, and the (n-component) spinor $\psi(x)$ is the Weyl symbol of $\psi(\hat{x})$ defined by [22]

$$\psi(x) = \frac{\sqrt{\det\theta}}{(2\pi)^2}\int d^4k e^{ikx}\mathrm{tr}(\psi(\hat{x})\hat{T}^\dagger(k)) \tag{2·2}$$

with $\hat{T}(k) = e^{ik_\mu\hat{x}^\mu}$ and $\hat{T}^\dagger(k) = \hat{T}(-k)$. The trace tr is taken in the Hilbert space in which the operators $\hat{x}^\mu$ are represented, and normalized [**] to give the last equality in Eq. (2·1).

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[*] By the commutative limit we always mean the limit $\theta^{\mu\nu}\to 0$ in the Lagrangian level.

[**] We shall prove the trace formula $\mathrm{tr}\hat{T}(k) = [(2\pi)^2/\sqrt{\det\theta}]\delta^4(k)$ in the Appendix A.

We then require the gauge invariance under the gauge transformation

$$\begin{cases} \psi(\hat{x}) \to {}^g\psi(\hat{x}) = g(\hat{x})\psi(\hat{x})u^\dagger(\hat{x}), \\ \bar{\psi}(\hat{x}) \to {}^g\bar{\psi}(\hat{x}) = u(\hat{x})\bar{\psi}(\hat{x})g^\dagger(\hat{x}), \end{cases} \quad g(\hat{x}) \in M_n(\mathcal{A}_x), \quad u(\hat{x}) \in M_1(\mathcal{A}_x), \tag{2·3}$$

with $g(\hat{x})g^\dagger(\hat{x}) = g^\dagger(\hat{x})g(\hat{x}) = \mathbf{1}_n$ and $u(\hat{x})u^\dagger(\hat{x}) = u^\dagger(\hat{x})u(\hat{x}) = \mathbf{1}$, where $\mathbf{1}_n$ is the n-dimensional unit-operator matrix and $M_n(\mathcal{A}_x)$ denotes the set of n-dimensional square matrices with elements in the algebra $\mathcal{A}_x$. The gauge invariance demands the replacement of the derivative $[\hat{p}_\mu, \psi(\hat{x})]$ in $\hat{S}_D$ with the covariant derivative,

$$[\hat{p}_\mu, \psi(\hat{x})] \to [\hat{p}_\mu, \psi(\hat{x})] + A_\mu(\hat{x})\psi(\hat{x}) - \psi(\hat{x})B_\mu(\hat{x}), \tag{2·4}$$

where the noncommutative gauge fields $A_\mu(\hat{x})$ and $B_\mu(\hat{x})$ are assumed to transform like

$$\begin{aligned} A_\mu(\hat{x}) &\to {}^gA_\mu(\hat{x}) = g(\hat{x})A_\mu(\hat{x})g^\dagger(\hat{x}) + g(\hat{x})[\hat{p}_\mu, g^\dagger(\hat{x})], \\ B_\mu(\hat{x}) &\to {}^gB_\mu(\hat{x}) = u(\hat{x})B_\mu(\hat{x})u^\dagger(\hat{x}) + u(\hat{x})[\hat{p}_\mu, u^\dagger(\hat{x})], \end{aligned} \tag{2·5}$$

or, equivalently, putting $A = i\gamma^\mu A_\mu$, $B = i\gamma^{\mu T} B_\mu$, $\hat{D}_0 = i\gamma^\mu \hat{p}_\mu$ and $\hat{D}_0^T = i\gamma^{\mu T}\hat{p}_\mu$, we have

$$\begin{aligned} A(\hat{x}) &\to {}^gA(\hat{x}) = g(\hat{x})A(\hat{x})g^\dagger(\hat{x}) + g(\hat{x})[\hat{D}_0, g^\dagger(\hat{x})], \\ B(\hat{x}) &\to {}^gB(\hat{x}) = u(\hat{x})B(\hat{x})u^\dagger(\hat{x}) + u(\hat{x})[\hat{D}_0^T, u^\dagger(\hat{x})]. \end{aligned} \tag{2·6}$$

The gauge-invariant, noncommutative Dirac action is thus obtained as

$$\hat{S}_{D+A-B} = (2\pi)^2\sqrt{\det\theta}\,\mathrm{tr}\bar{\psi}(\hat{x})(i\gamma^\mu[\hat{p}_\mu, \psi(\hat{x})] + A(\hat{x})\psi(\hat{x}) - \psi(\hat{x})B(\hat{x}) - M\psi(\hat{x})), \tag{2·7}$$

where we have assumed that M is gauge-invariant.

Since $\hat{p}_\mu$ is anti-hermitian, so is $A_\mu(\hat{x})$, $A_\mu^\dagger(\hat{x}) = -A_\mu(\hat{x})$ and similarly for $B_\mu(\hat{x})$, ensuring the hermiticity of $\hat{S}_{D+A-B}$. The noncommutative field strengths

$$\begin{aligned} F_{\mu\nu}(\hat{x}) &= [\hat{p}_\mu, A_\nu(\hat{x})] - [\hat{p}_\nu, A_\mu(\hat{x})] + [A_\mu(\hat{x}), A_\nu(\hat{x})], \\ G_{\mu\nu}(\hat{x}) &= [\hat{p}_\mu, B_\nu(\hat{x})] - [\hat{p}_\nu, B_\mu(\hat{x})] + [B_\mu(\hat{x}), B_\nu(\hat{x})], \end{aligned} \tag{2·8}$$

are also anti-hermitian. Since $[\hat{p}_\mu, \hat{p}_\nu] = i\theta_{\mu\nu}$ commutes with $\hat{x}^\rho$, the field strengths are gauge-covariant

$$\begin{aligned} F_{\mu\nu}(\hat{x}) &\to {}^gF_{\mu\nu}(\hat{x}) = g(\hat{x})F_{\mu\nu}(\hat{x})g^\dagger(\hat{x}), \\ G_{\mu\nu}(\hat{x}) &\to {}^gG_{\mu\nu}(\hat{x}) = u(\hat{x})G_{\mu\nu}(\hat{x})u^\dagger(\hat{x}). \end{aligned} \tag{2·9}$$

Consequently, the noncommutative Yang-Mills (NCYM) action is given by

$$\hat{S}_{YM} = -\frac{1}{2}(2\pi)^2\sqrt{\det\theta}\mathrm{Tr}\frac{1}{g^2}F^{\dagger}_{\mu\nu}(\hat{x})F^{\mu\nu}(\hat{x}) - \frac{1}{2g'^2}(2\pi)^2\sqrt{\det\theta}\mathrm{tr}G^{\dagger}_{\mu\nu}(\hat{x})G^{\mu\nu}(\hat{x}), \tag{2·10}$$

where Tr includes the trace over the internal symmetry matrices in addition to the previously-defined trace tr and $F^{\mu\nu}(\hat{x}) = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma}(\hat{x})$. We should delete the second term in the above equation if the gauge field B_μ appears already in $F_{\mu\nu}$ in order to avoid the double counting.

Since the determinant for the operator-valued gauge function $g(\hat{x})$ can not be well-defined, we can formulate only noncommutative $U(2)$ but not noncommutative $SU(2)$. (We may extend $2 \to N$.) Moreover, the commutative limit of noncommutative $U(2)$ is $U(1) \times SU(2)$ YM with the same coupling constant. In order to recover $U(1) \times SU(2)$ YM with the different coupling constants it is preferable to consider noncommutative $U(2)$ which is reduced to $SU(2)$ YM in the commutative limit, plus additional noncommutative $U(1)^2$ (with the same coupling constant) reduced to commutative $U(1)$. In such noncommutative $U(2)$ an Abelian gauge field mixed with the non-Abelian gauge fields on noncommutative space-times would 'disappear' in the commutative limit because it is proportional to θ for small θ, while the non-Abelian gauge fields exist for $\theta \to 0$. If such a model is possible, it will serve to define a noncommutative GWS model which is reduced to the usual GWS theory in the commutative limit. We shall argue below that a noncommutative Connes' YM may play a role in this direction.

To define a noncommutative Connes' YM we consider [19] the 'gauge' transformations

$$\begin{cases} \psi(\hat{x}) \to b_i(\hat{x})\psi(\hat{x})c_i^{\dagger}(\hat{x}), \\ \bar{\psi}(\hat{x}) \to d_i^{\dagger}(\hat{x})\bar{\psi}(\hat{x})a_i(\hat{x}), \end{cases} \qquad a_i(\hat{x}), b_i(\hat{x}) \in M_n(\mathcal{A}_x), \quad c_i(\hat{x}), d_i(\hat{x}) \in M_1(\mathcal{A}_x) \tag{2·11}$$

with

$$\sum_i a_i(\hat{x})b_i(\hat{x}) = \mathbf{1}_n, \quad \sum_i c_i^{\dagger}(\hat{x})d_i^{\dagger}(\hat{x}) = \mathbf{1}, \tag{2·12}$$

to obtain after taking the sum over the index i in constructing the sensible action the gauge fields $A(\hat{x})$and $B(\hat{x})$ in Eq. (2·7) as the sums

$$A(\hat{x}) = \sum_i a_i(\hat{x})[\hat{D}_0, b_i(\hat{x})], \quad B(\hat{x}) = \sum_i c_i^{\dagger}(\hat{x})[\hat{D}_0^T, d_i^{\dagger}(\hat{x})]. \tag{2·13}$$

Equation (2·13) is similar to Connes' expression for YM gauge field. In fact, in the commutative limit, we may replace $\hat{x}^\mu \to x^\mu$ and $\hat{D}_0 \to D_0$, obtaining the noncommutative one-form on M_4.

We define the field strength by the wedge product [19] of the Dirac matrices

$$F(\hat{x}) = \sum_i [\hat{D}_0, a_i(\hat{x})] \wedge [\hat{D}_0, b_i(\hat{x})] + A(\hat{x}) \wedge A(\hat{x}) = -\frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)F_{\mu\nu}(\hat{x}),$$

$$G(\hat{x}) = \sum_i [\hat{D}_0^T, c_i^\dagger(\hat{x})] \wedge [\hat{D}_0^T, d_i^\dagger(\hat{x})] + B(\hat{x}) \wedge B(\hat{x}) = -\frac{1}{4}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu)^T G_{\mu\nu}(\hat{x}), \quad (2\cdot14)$$

where $F_{\mu\nu}(\hat{x})$ and $G_{\mu\nu}(\hat{x})$ are given by Eq. (2·8) with $A_\mu(\hat{x}) = \sum_i a_i(\hat{x})[\hat{p}_\mu, b_i(\hat{x})]$ and $B_\mu(\hat{x}) = \sum_i c_i^\dagger(\hat{x})[\hat{p}_\mu, d_i^\dagger(\hat{x})]$. NCYM action (2·10) then reads

$$\hat{S}_{YM} = -\frac{1}{4g^2}(2\pi)^2\sqrt{\det\theta}\,\boldsymbol{Tr}F(\hat{x})F(\hat{x}) - \frac{1}{4g'^2}(2\pi)^2\sqrt{\det\theta}\,\boldsymbol{tr}G(\hat{x})G(\hat{x}), \quad (2\cdot15)$$

where $\boldsymbol{Tr}$ and $\boldsymbol{tr}$ includes the trace over the Dirac matrices as well. The theory defined by the sum $\hat{S}_{D+A-B} + \hat{S}_{YM}$ involves only the physical fields.

If M is not gauge-invariant and fermions exist in chiral multiplets, we use the chiral decomposition of spinors so that the Dirac operator reads

$$D = D_0 + i\gamma_5 M,\ D_0 = \begin{pmatrix} i\partial\!\!\!/ \otimes 1_{n_L} & 0 \\ 0 & i\partial\!\!\!/ \otimes 1_{n_R} \end{pmatrix} \otimes 1_{N_g},\ M = \begin{pmatrix} 0 & M_1 \\ M_1^\dagger & 0 \end{pmatrix}, \quad (2\cdot16)$$

with N_g being the number of generations. The γ_5 matrix is inserted for later convenience. The 'gauge' transformations (2·11) except for $c_i^\dagger(\hat{x})$ and $d_i^\dagger(\hat{x})$ are to be extended to those of 2×2 matrices in the chiral space

$$f_i(\hat{x}) = \begin{pmatrix} f_i^L(\hat{x}) & 0 \\ 0 & f_i^R(\hat{x}) \end{pmatrix} \otimes 1_{N_g},\ f_i^L(\hat{x}) \in M_{n_L}(\mathcal{A}_x), \quad f_i^R(\hat{x}) \in M_{n_R}(\mathcal{A}_x),\ f = a, b. \quad (2\cdot17)$$

The same procedure as described for the case of the gauge-invariant M leads to the generalized noncommutative gauge field

$$\boldsymbol{A}(\hat{x}) = \sum_i a_i(\hat{x})[\hat{D}, b_i(\hat{x})] = A(\hat{x}) + i\gamma_5\Phi(\hat{x}),\ \Phi(\hat{x}) = \sum_i a_i(\hat{x})[M, b_i(\hat{x})], \quad (2\cdot18)$$

where $\hat{D} = \hat{D}_0 + i\gamma_5 M\mathbf{1}$ and $A(\hat{x}) = \sum_i a_i(\hat{x})[\hat{D}_0, b_i(\hat{x})] = \begin{pmatrix} A^L(\hat{x}) & 0 \\ 0 & A^R(\hat{x}) \end{pmatrix} \otimes 1_{N_g}$. The gauge field $B(\hat{x})$ remains the same as before. The fields $\boldsymbol{A}(\hat{x})$ and $B(\hat{x})$ appear in the noncommutative Dirac-Yukawa action

$$\begin{aligned}\hat{S}_{\boldsymbol{D}} &= (2\pi)^2\sqrt{\det\theta}\,\mathrm{tr}\bar{\psi}(\hat{x})(i\gamma^\mu[\hat{p}_\mu, \psi(\hat{x})] + \boldsymbol{A}(\hat{x})\psi(\hat{x}) - \psi(\hat{x})B(\hat{x}) + i\gamma_5 M\psi(\hat{x})) \\ &= (2\pi)^2\sqrt{\det\theta}\,\mathrm{tr}\bar{\psi}(\hat{x})(i\gamma^\mu[\hat{p}_\mu, \psi(\hat{x})] + A(\hat{x})\psi(\hat{x}) - \psi(\hat{x})B(\hat{x}) + i\gamma_5 H(\hat{x})\psi(\hat{x})) \quad (2\cdot19)\end{aligned}$$

with $H(\hat{x}) = \Phi(\hat{x}) + M\mathbf{1}$.

The gauge transformation

$$\boldsymbol{A}(\hat{x}) \to {}^g\boldsymbol{A}(\hat{x}) = g(\hat{x})\boldsymbol{A}(\hat{x})g^\dagger(\hat{x}) + g(\hat{x})[\hat{D}, g^\dagger(\hat{x})] \tag{2·20}$$

is induced by $b_i(\hat{x}) \to b_i(\hat{x})g^\dagger(\hat{x})$ and $a_i(\hat{x}) \to g(\hat{x})a_i(\hat{x})$, where

$$g(\hat{x}) = \begin{pmatrix} g_L(\hat{x}) & 0 \\ 0 & g_R(\hat{x}) \end{pmatrix} \otimes 1_{N_g}, \quad g_L(\hat{x}) \in M_{n_L}(\mathcal{A}_x), \quad g_R(\hat{x}) \in M_{n_R}(\mathcal{A}_x), \tag{2·21}$$

with the conditions $g_L(\hat{x})g_L^\dagger(\hat{x}) = g_L^\dagger(\hat{x})g_L(\hat{x}) = \mathbf{1}_{n_L}$ and $g_R(\hat{x})g_R^\dagger(\hat{x}) = g_R^\dagger(\hat{x})g_R(\hat{x}) = \mathbf{1}_{n_R}$.

In order to construct the bosonic action we again employ the wedge product [19] of the Dirac matrices to define the generalized noncommutative field strength

$$\boldsymbol{F}(\hat{x}) = \sum_i [\hat{D}, a_i(\hat{x})] \wedge [\hat{D}, b_i(\hat{x})] + \boldsymbol{A}(\hat{x}) \wedge \boldsymbol{A}(\hat{x}) = F(\hat{x}) - i\gamma_5[\hat{P}, H(\hat{x})] - 1_4 \otimes Y_0(\hat{x}), \tag{2·22}$$

where $\hat{P} = i\gamma^\mu \hat{P}_\mu$ with $\hat{P}_\mu = \hat{p}_\mu + A_\mu$, and

$$Y_0(\hat{x}) = H^2(\hat{x}) - M^2 + y(\hat{x}), \quad y(\hat{x}) \equiv -\sum_i a_i(\hat{x})[M^2, b_i(\hat{x})]. \tag{2·23}$$

Unfortunately, however, there is a nuisance in this definition because $\boldsymbol{F}(\hat{x})$ does not vanish even when $\boldsymbol{A}(\hat{x}) = \sum_i a_i(\hat{x})[\hat{D}, b_i(\hat{x})] = 0$. This is a common feature [1), 2), 3), 4), 5)] in Connes' YM, which arises from the ambiguity in defining the exterior derivative as given by the first term in Eq. (2·22) based on the sum (2·18).

To overcome the difficulty we resort to a subtraction method similar to Connes' one [1), 2), 3), 4), 5)] of introducing a quotient algebra. It consists of subtracting off the piece $\langle \boldsymbol{F}(\hat{x}) \rangle$, which is a matrix of the *same form* [*)] as $\sum_i [\hat{D}, a_i(\hat{x})] \wedge [\hat{D}, b_i(\hat{x})]$ with $\boldsymbol{A}(\hat{x}) = \sum_i a_i(\hat{x})[\hat{D}, b_i(\hat{x})] = 0$, from $\boldsymbol{F}(\hat{x})$. The genuine noncommutative generalized field strength is then given by $[\boldsymbol{F}(\hat{x})] = \boldsymbol{F}(\hat{x}) - \langle \boldsymbol{F}(\hat{x}) \rangle$. Since $\sum_i [\hat{D}, a_i(\hat{x})] \wedge [\hat{D}, b_i(\hat{x})]|_{\boldsymbol{A}(\hat{x})=0} = -1_4 \otimes y(\hat{x})$, we have $\langle \boldsymbol{F}(\hat{x}) \rangle = -1_4 \otimes \langle Y_0(\hat{x}) \rangle$ where $\langle Y_0(\hat{x}) \rangle$ is a matrix of the *same form* as $y(\hat{x})$. Consequently, we obtain

$$[\boldsymbol{F}(\hat{x})] = F(\hat{x}) - i\gamma_5[\hat{P}, H(\hat{x})] - 1_4 \otimes [Y_0(\hat{x})], \quad [Y_0(\hat{x})] = Y_0(\hat{x}) - \langle Y_0(\hat{x}) \rangle, \tag{2·24}$$

[*)] For instance, a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is of the same form as $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ if both are hermitian. The subtracted piece is uniquely determined by the orthogonality.

leading to the noncommutative Yang-Mills-Higgs (NCYMH) action

$$
\begin{aligned}
\hat{S}_{YMH} &= -\frac{1}{4N_g}(2\pi)^2\sqrt{\det\theta}\,\boldsymbol{Tr}_{cg}\frac{1}{g^2}[\boldsymbol{F}(\hat{x})][\boldsymbol{F}(\hat{x})] - \frac{1}{4g'^2}(2\pi)^2\sqrt{\det\theta}\,\boldsymbol{tr}G(\hat{x})G(\hat{x}) \\
&= \hat{S}_{YM} + \frac{1}{N_g}(2\pi)^2\sqrt{\det\theta}\,\mathrm{Tr}_{cg}\frac{1}{g^2}[\hat{P}_\mu, H(\hat{x})][\hat{P}^\mu, H(\hat{x})] \\
&\quad -\frac{1}{2N_g}(2\pi)^2\sqrt{\det\theta}\,\mathrm{Tr}_{cg}\frac{1}{g^2}[Y_0(\hat{x})]^2,
\end{aligned}
\tag{2·25}
$$

where the subscripts c and g of $\boldsymbol{Tr}_{cg}$ and Tr_{cg} indicate the traces in the chiral and generation spaces, respectively, and $\hat{P}^\mu = g^{\mu\nu}\hat{P}_\nu$.

It is necessary to fix the model in order to make the subtraction $[Y_0(\hat{x})] = Y_0(\hat{x}) - \langle Y_0(\hat{x})\rangle$. A noncommutative GWS model in the leptonic sector is obtained by taking $n_L = n_R = 2$ with $M_{n_L=2}(\mathcal{A}_x) \to \boldsymbol{H}(\mathcal{A}_x)$ and $M_{n_R=2}(\mathcal{A}_x) \to \boldsymbol{B}(\mathcal{A}_x)$, where

$$
\begin{pmatrix} \alpha(\hat{x}) & \beta(\hat{x}) \\ -\beta^\dagger(\hat{x}) & \alpha^\dagger(\hat{x}) \end{pmatrix} \in \boldsymbol{H}(\mathcal{A}_x), \quad \begin{pmatrix} b(\hat{x}) & 0 \\ 0 & b^\dagger(\hat{x}) \end{pmatrix} \in \boldsymbol{B}(\mathcal{A}_x)
$$

with $\alpha(\hat{x}), \beta(\hat{x}), b(\hat{x}) \in M_1(\mathcal{A}_x)$. In this model the left-handed fermions are doublets like $\begin{pmatrix} \nu \\ e \end{pmatrix}_L$ and the right-handed fermions singlets like $\begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$ in N_g generations with the mass matrix

$$
M_1 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad m_{1,2}:\ N_g \times N_g \text{ matrices.}
$$

It is then straightforward to show that

$$
H(\hat{x}) = \begin{pmatrix} 0 & h(\hat{x})M_1 \\ M_1^\dagger h^\dagger(\hat{x}) & 0 \end{pmatrix}, \quad h(\hat{x}) = \begin{pmatrix} {\phi'_0}^\dagger(\hat{x}) & \phi_+(\hat{x}) \\ -{\phi'_+}^\dagger(\hat{x}) & \phi_0(\hat{x}) \end{pmatrix}. \tag{2·26}
$$

The two Higgs doublets

$$
\phi(\hat{x}) = \begin{pmatrix} \phi_+(\hat{x}) \\ \phi_0(\hat{x}) \end{pmatrix}, \quad \phi^c(\hat{x}) = \begin{pmatrix} {\phi'_0}^\dagger(\hat{x}) \\ -{\phi'_+}^\dagger(\hat{x}) \end{pmatrix} \tag{2·27}
$$

fuse into a single Higgs doublet in the commutative limit since the operators defining them become commutative in that limit *). It follows from Eq. (2·20) that, under the gauge transformation by

*) In the commutative limit $\phi^c(\hat{x}) \to \phi^c(x)$ and $\phi(\hat{x}) \to \phi(x)$ with $\phi^c(x) = i\sigma_2\phi^*(x)$ in terms of the second Pauli matrix σ_2. The change of the spectrum is characteristic to our formulation of a noncommutative GWS model which is reduced to the GWS theory in the commutative limit.

$g_L(\hat{x}) \in \boldsymbol{H}(\mathcal{A}_x)$ and $g_R(\hat{x}) \in \boldsymbol{B}(\mathcal{A}_x)$ with the conditions $g_L(\hat{x})g_L^\dagger(\hat{x}) = g_R(\hat{x})g_R^\dagger(\hat{x}) = \mathbf{1}_2$, $h(\hat{x})$ transforms as

$$h(\hat{x}) \rightarrow^g h(\hat{x}) = g_L(\hat{x})h(\hat{x})g_R^\dagger(\hat{x}). \tag{2·28}$$

On the other hand, the gauge transformation, $\psi(\hat{x}) \rightarrow g(\hat{x})\psi(\hat{x})u^\dagger(\hat{x})$, for the chiral leptons gets factorized in the commutative limit into two factors [18] [*)].

It can be shown that $y(\hat{x}) = \begin{pmatrix} y_1(\hat{x}) & 0 \\ 0 & 0 \end{pmatrix}$, where $y_1(\hat{x})$ is a hermitian matrix. On the other hand, $H^2(\hat{x}) - M^2 = \begin{pmatrix} y_1'(\hat{x}) & 0 \\ 0 & y_2(\hat{x}) \end{pmatrix}$, where $y_1'(\hat{x})$ is also a hermitian matrix not orthogonal to $y_1(\hat{x})$, and $y_2(\hat{x})$ is given by

$$y_2(\hat{x}) = \begin{pmatrix} (\phi^{c\dagger}(\hat{x})\phi^c(\hat{x}) - \mathbf{1})m_1^\dagger m_1 & \phi^{c\dagger}(\hat{x})\phi(\hat{x})m_1^\dagger m_2 \\ \phi^\dagger(\hat{x})\phi^c(\hat{x})m_2^\dagger m_1 & (\phi^\dagger(\hat{x})\phi(\hat{x}) - \mathbf{1})m_2^\dagger m_2 \end{pmatrix}.$$

The result of the subtraction is $[Y_0(\hat{x})] = \begin{pmatrix} 0 & 0 \\ 0 & y_2(\hat{x}) \end{pmatrix}$. After rescaling NCYMH action reads

$$\begin{aligned} \hat{S}_{YMH} = \hat{S}_{YM} &+ \frac{1}{2}(2\pi)^2\sqrt{\det\theta}\mathrm{Tr}_c[\hat{D}_\mu, h(\hat{x})]^\dagger[\hat{D}^\mu, h(\hat{x})] \\ &- \frac{\lambda'}{4}(2\pi)^2\sqrt{\det\theta}\mathrm{tr}[(\phi^{c\dagger}(\hat{x})\phi^c(\hat{x}) - \frac{v^2}{2}\mathbf{1})^2\mathrm{tr}_g(m_1^\dagger m_1)^2 \\ &+ \phi^{c\dagger}(\hat{x})\phi(\hat{x})\phi^\dagger(\hat{x})\phi^c(\hat{x})\mathrm{tr}_g(m_1 m_1^\dagger m_2 m_2^\dagger)] \\ &- \frac{\lambda'}{4}(2\pi)^2\sqrt{\det\theta}\mathrm{tr}[(\phi^\dagger(\hat{x})\phi(\hat{x}) - \frac{v^2}{2}\mathbf{1})^2\mathrm{tr}_g(m_2^\dagger m_2)^2 \\ &+ \phi^\dagger(\hat{x})\phi^c(\hat{x})\phi^{c\dagger}(\hat{x})\phi(\hat{x})\mathrm{tr}_g(m_1 m_1^\dagger m_2 m_2^\dagger)] \end{aligned} \tag{2·29}$$

with $[\hat{D}_\mu, h(\hat{x})] = [\hat{p}_\mu, h(\hat{x})] + A_\mu^L(\hat{x})h(\hat{x}) - h(\hat{x})A_\mu^R(\hat{x})$, $\hat{D}^\mu = g^{\mu\nu}\hat{D}_\nu$ and tr_g meaning the trace in the generation space. The parameters v^2, λ' are expressed in terms of the gauge coupling constants, N_g and the generation-space traces of the matrices $m_{1,2}$.

In NCYMH action (2·29) we are left with only the physical degrees of freedom, $A_\mu^{L,R}(\hat{x})$, $B_\mu(\hat{x})$, $\phi(\hat{x})$ and $\phi^c(\hat{x})$. We now turn to study the Higgs mechanism on noncommutative space-times.

[*)] It should be remembered that the factorization of the gauge transformations in Connes' scheme is required to reproduce the correct hypercharge of leptons using the doubled spinor [18] in accord with Connes' real structure [3]. Here we do not have to introduce the doubled spinor in order to obtain the correct charge assignment.

§3. Noncommutative GWS model in the leptonic sector

Since the Higgs mechanism in our noncommutative GWS model becomes most transparent in the Weyl-Moyal description of the noncommutative Connes' YM, we shall first translate the operator language into the function-space language with deformed product.

Using the relation $\hat{T}(k)\hat{T}(k') = e^{-\frac{i}{2}k_{1\mu}\theta^{\mu\nu}k_{2\nu}}\hat{T}(k+k')$ together with (see Eq. (2·2))

$$\varphi(\hat{x}) = \frac{1}{(2\pi)^4}\int d^4k d^4x \varphi(x) e^{-ikx}\hat{T}(k),$$

we find [22] the basic formulae of the translation

$$\frac{\sqrt{\det\theta}}{(2\pi)^2}\int d^4k e^{ikx}\mathrm{tr}(\varphi_1(\hat{x})\varphi_2(\hat{x})\hat{T}^\dagger(k)) = \varphi_1(x) * \varphi_2(x),$$

$$\frac{\sqrt{\det\theta}}{(2\pi)^2}\int d^4k e^{ikx}\mathrm{tr}(\varphi_1(\hat{x})\varphi_2(\hat{x})\varphi_3(\hat{x})\hat{T}^\dagger(k)) = \varphi_1(x) * \varphi_2(x) * \varphi_3(x), \tag{3·1}$$

where the $*$ product is the Moyal product,

$$\varphi_1(x) * \varphi_2(x) = e^{\frac{i}{2}\frac{\partial}{\partial x_1^\mu}\theta^{\mu\nu}\frac{\partial}{\partial x_2^\nu}}\varphi_1(x_1)\varphi_2(x_2)\Big|_{x_1=x_2=x}.$$

Integration gives

$$(2\pi)^2\sqrt{\det\theta}\mathrm{tr}(\varphi_1(\hat{x})\varphi_2(\hat{x})) = \int d^4x \varphi_1(x) * \varphi_2(x) = \int d^4x \varphi_1(x)\varphi_2(x),$$

$$(2\pi)^2\sqrt{\det\theta}\mathrm{tr}(\varphi_1(\hat{x})\varphi_2(\hat{x})\varphi_3(\hat{x})) = \int d^4x \varphi_1(x) * \varphi_2(x) * \varphi_3(x). \tag{3·2}$$

Using these formulae we rewrite the 'gauge' transformations (2·11) as

$$\begin{cases} \psi(x) \to b_i(x) * \psi(x) * c_i^\dagger(x), \\ \bar{\psi}(x) \to d_i^\dagger(x) * \bar{\psi}(x) * a_i(x), \end{cases} \tag{3·3}$$

where the gauge parameters $f_i(x) = \begin{pmatrix} f_i^L(x) & 0 \\ 0 & f_i^R(x) \end{pmatrix} \otimes 1_{N_g}, \begin{cases} f_i^L(x) \in C^\infty(M_4) \otimes M_{n_L}(\boldsymbol{C}) \\ f_i^R(x) \in C^\infty(M_4) \otimes M_{n_R}(\boldsymbol{C}) \end{cases}$, $f = a, b$ and $c_i(x), d_i(x) \in C^\infty(M_4) \otimes M_1(\boldsymbol{C})$ satisfy $\sum_i a_i(x) * b_i(x) = 1_n, n = N_g(n_L + n_R)$ and $\sum_i c_i^\dagger(x) * d_i^\dagger(x) = 1$. The gauge fields are given by

$$\begin{cases} \boldsymbol{A}(x) = \sum_i a_i(x) * [D, b_i(x)] = A(x) + i\gamma_5 \Phi(x), \\ B(x) = \sum_i c_i^\dagger(x) * [D_0^T, d_i^\dagger(x)], \quad D_0^T = i\gamma^{\mu T}\partial_\mu, \end{cases} \tag{3·4}$$

with $A(x) = \sum_i a_i(x) * [D_0, b_i(x)] = \begin{pmatrix} A^L(x) & 0 \\ 0 & A^R(x) \end{pmatrix} \otimes 1_{N_g}$. The noncommutative Dirac-Yukawa action (2·19) is brought into the form (before rescaling of $H = \Phi(x) + M$)

$$\hat{S}_{\boldsymbol{D}} = \int d^4x \bar{\psi}(x)(D\psi(x) + *\boldsymbol{A}(x) * \psi(x) - *\psi(x) * B(x) + i\gamma_5 M\psi(x))$$
$$= \int d^4x \bar{\psi}(x)(D_0\psi(x) + *A(x) * \psi(x) - *\psi(x) * B(x) + i\gamma_5 * H(x) * \psi(x)). \quad (3·5)$$

It is gauge-invariant under

$$\begin{cases} \psi(x) \to g(x) * \psi(x) * U^\dagger(x), \\ \bar{\psi}(x) \to U(x) * \bar{\psi}(x) * g^\dagger(x), \end{cases}$$
$$\begin{cases} \boldsymbol{A}(x) \to^g \boldsymbol{A}(x) = g(x) * \boldsymbol{A}(x) * g^\dagger(x) + g(x) * [D, g^\dagger(x)], \\ B(x) \to^g B(x) = U(x) * B(x) * U^\dagger(x) + U(x) * [D_0^T, U^\dagger(x)], \end{cases} \quad (3·6)$$

with

$$g(x) = \begin{pmatrix} g_L(x) & 0 \\ 0 & g_R(x) \end{pmatrix} \otimes 1_{N_g},$$
$$g_L(x) \in C^\infty(M_4) \otimes M_{n_L}(\boldsymbol{C}), \quad g_R(\hat{x}) \in C^\infty(M_4) \otimes M_{n_R}(\boldsymbol{C}),$$
$$g(x) * g^\dagger(x) = 1_n, \quad n = N_g(n_L + n_R), \quad U(x) * U^\dagger(x) = 1. \quad (3·7)$$

Let us next turn to the bosonic sector. The previous model amounts to replace $M_{n_L=2}(\boldsymbol{C}) \to \boldsymbol{H}$ and $M_{n_R=2}(\boldsymbol{C}) \to \boldsymbol{B}$ *). The YM sector is well-known. The Higgs kinetic energy term in Eq. (2·29) is converted into

$$\hat{S}_{HK} \equiv \frac{1}{2}(2\pi)^2\sqrt{\det\theta}\mathrm{Tr}_c[\hat{D}_\mu, h(\hat{x})]^\dagger[\hat{D}^\mu, h(\hat{x})] = \frac{1}{2}\int d^4x \mathrm{tr}_c\{D_\mu, h(x)\}_M^\dagger * \{D^\mu, h(x)\}_M, \quad (3·8)$$

with $\{D_\mu, h(x)\}_M = \partial_\mu h(x) + A_\mu^L(x) * h(x) - h(x) * A_\mu^R(x)$. The tr_c indicates the trace in the chiral space. Putting $\hat{S}_{YMH} = \hat{S}_{YM} + \hat{S}_{HK} + \hat{S}_{HP}$ we have the Higgs ‘potential’ term

$$-\hat{S}_{HP} = \int d^4x \Big(\frac{\lambda'}{4}[(\phi^{c\dagger}(x) * \phi^c(x) - \frac{v^2}{2}) * (\phi^{c\dagger}(x) * \phi^c(x) - \frac{v^2}{2})\mathrm{tr}_g(m_1^\dagger m_1)^2$$
$$+\phi^{c\dagger}(x) * \phi(x) * \phi^\dagger(x) * \phi^c(x)\mathrm{tr}_g(m_1 m_1^\dagger m_2 m_2^\dagger)]$$
$$+\frac{\lambda'}{4}[(\phi^\dagger(x) * \phi(x) - \frac{v^2}{2}) * (\phi^\dagger(x) * \phi(x) - \frac{v^2}{2})\mathrm{tr}_g(m_2^\dagger m_2)^2$$
$$+\phi^\dagger(x) * \phi^c(x) * \phi^{c\dagger}(x) * \phi(x)\mathrm{tr}_g(m_1 m_1^\dagger m_2 m_2^\dagger)]\Big). \quad (3·9)$$

*) Here, $\boldsymbol{H}$ is the real quaternions and $\boldsymbol{B} \subset \boldsymbol{H}$ is the set of elements $\begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix}$ for $b \in \boldsymbol{C}$.

We find that in the commutative limit the integrand is reduced to the usual Higgs potential for a single Higgs doublet.

The Higgs mechanism occurs if a minimum of $-\hat{S}_{HP}$ is attained by non-vanishing vacuum expectation value (VEV) $\langle\phi(x)\rangle$ of the Higgs field $\phi(x)$. We seek for the minimum by assuming that the VEV is constant, $\langle\phi(x)\rangle \equiv \langle\phi\rangle$, and $\langle\phi^c\rangle = i\sigma_2\langle\phi\rangle^*$. In this case the coefficients of $\mathrm{tr}_g(m_1 m_1^\dagger m_2 m_2^\dagger)$ vanish *). The rest is minimized if

$$\langle\phi^\dagger(x) * \phi(x)\rangle = \langle\phi^\dagger\rangle\langle\phi\rangle = \frac{v^2}{2}, \quad \langle\phi^{c\dagger}(x) * \phi^c(x)\rangle = \langle\phi^{c\dagger}\rangle\langle\phi^c\rangle = \frac{v^2}{2}. \tag{3·10}$$

The gauge transformation for Higgs doublets is given by

$$\phi(x) \to {}^g\phi(x) = g_L(x) * \phi(x) * U(x), \quad \phi^c(x) \to {}^g\phi^c(x) = g_L(x) * \phi(x)^c * U^\dagger(x), \tag{3·11}$$

where $g_L(x) \in C^\infty(M_4) \otimes \boldsymbol{H}$ with $g_L(x) * g_L^\dagger(x) = 1_2$, while $g_R(x) = \begin{pmatrix} U(x) & 0 \\ 0 & U^\dagger(x) \end{pmatrix} \in C^\infty(M_4) \otimes \boldsymbol{B}$ with $U(x) * U^\dagger(x) = 1$. Remember that the same function $U(x)$ as in Eq. (3·6) appears also in $g_R(x)$. Consequently, we should retain only the first term in Eq. (2·15) to define the YM action $\hat{S}_{YM}$. We assume the unbroken symmetry **)

$$\begin{aligned} \langle\phi\rangle \to \langle{}^h\phi\rangle &= h_L(x) * \langle\phi\rangle * U(x) = h_L(x) * U(x)\langle\phi\rangle = \langle\phi\rangle, \\ \langle\phi^c\rangle \to \langle{}^h\phi^c\rangle &= h_L(x) * \langle\phi^c\rangle * U^\dagger(x) = h_L(x) * U^\dagger(x)\langle\phi^c\rangle = \langle\phi^c\rangle. \end{aligned} \tag{3·12}$$

This together with Eq. (3·10) has a solution

$$h_L(x) = g_R(x) = \begin{pmatrix} U(x) & 0 \\ 0 & U^\dagger(x) \end{pmatrix} \in C^\infty(M_4) \otimes \boldsymbol{B},$$

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \langle\phi_0\rangle = \dfrac{v}{\sqrt{2}} \end{pmatrix}, \quad \langle\phi^c\rangle = \begin{pmatrix} \langle{\phi'}_0^\dagger\rangle = \dfrac{v}{\sqrt{2}} \\ 0 \end{pmatrix}. \tag{3·13}$$

The unbroken symmetry for leptons is given by

$$\begin{cases} \nu(x) \to U(x) * \nu(x) * U^\dagger(x), \\ e(x) \to U^\dagger(x) * e(x) * U^\dagger(x). \end{cases} \tag{3·14}$$

It can be shown that we are left with two neutral and one charged massive Higgses among which only one neutral massive Higgs to be identified with the standard Higgs remains in the

*) For instance, $\langle(\phi^{c\dagger}(x) * \phi(x) * \phi^\dagger(x) * \phi^c(x)\rangle = \langle\phi^{c\dagger}\rangle\langle\phi\rangle\langle\phi^\dagger\rangle\langle\phi^c\rangle = 0$ provided that $\langle\phi^c\rangle = i\sigma_2\langle\phi\rangle^*$.

**) This assumption is motivated by generating the input fermion mass by the Higgs mechanism.

commutative limit.

We finally investigate the generation of the gauge boson masses. Remembering Eqs. (3·4) and (3·5) we put *)

$$A^L_\mu(x) = -\frac{ig}{2}\begin{pmatrix} A^0_\mu + A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & A^0_\mu - A^3_\mu \end{pmatrix}(x),$$

$$A^R_\mu(x) = -\frac{ig'}{2}\begin{pmatrix} B_\mu(x) & 0 \\ 0 & -C_\mu(x) \end{pmatrix}, \tag{3·15}$$

and rescale $B_\mu(x) \to -(ig'/2)B_\mu(x)$ in Eq. (3·5). In the commutative limit we have $A^0_\mu(x) \to 0$ and $C_\mu(x) \to B_\mu(x)$ **). Namely, the gauge field $A^L_\mu(x)$ is for noncommutative $U(2)$ reduced to commutative $SU(2)$. Similarly, the gauge field $A^R_\mu(x)$ is for noncommutative $U(1)^2$ (with the same coupling constant) reduced to commutative $U(1)$. Consequently, we have two different coupling constants in the commutative limit as desired for the commutative GWS theory. Setting

$$h(x) \to \langle h \rangle = \frac{v}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\hat{S}_{HK}$ is reduced to the x-integral of the mass terms

$$\frac{1}{2}\int d^4x \mathrm{tr}_c\{D_\mu, \langle h \rangle\}^\dagger_M * \{D_\mu, \langle h \rangle\}_M = \int d^4x[\frac{1}{2}M^2_W(W^\dagger_\mu(x)W^\mu(x) + W^\mu(x)W^\dagger_\mu(x)) + \frac{1}{4}M^2_Z(Z_\mu(x)Z^\mu(x) + {Z'}_\mu(x){Z'}^\mu(x))],$$

where $M^2_W = v^2g^2/4, M^2_Z = v^2(g^2 + g'^2)/4$ and

$$W_\mu = \frac{1}{\sqrt{2}}(A^1_\mu - iA^2_\mu),$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g(A^0_\mu + A^3_\mu) - g'B_\mu),$$

$${Z'}_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g(A^0_\mu - A^3_\mu) + g'C_\mu). \tag{3·16}$$

*) Both $A^L_\mu(x) = \sum_i a^L_i(x) * \partial_\mu b^L_i(x)$ and $A^R_\mu(x) = \sum_i a^R_i(x) * \partial_\mu b^R_i(x)$ are *not* traceless in contrast to the model in Ref.18).

**) The proof will be given in the Appendix B.

The orthogonal combinations

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'(A^0_\mu + A^3_\mu) + gB_\mu),$$

$$A'_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'(A^0_\mu - A^3_\mu) - gC_\mu) \tag{3·17}$$

remain massless, although $A'_\mu \to -A_\mu$ in the commutative limit.

The unbroken gauge transformation for mass-eigenstates gauge fields turns out to be

$$\begin{aligned}
{}^hW_\mu(x) &= U(x) * W_\mu(x) * U(x), \\
{}^hZ_\mu(x) &= U(x) * Z_\mu(x) * U^\dagger(x), \\
{}^hZ'_\mu(x) &= U^\dagger(x) * Z'_\mu(x) * U(x), \\
{}^hA_\mu(x) &= U(x) * A_\mu(x) * U^\dagger(x) + \frac{2i}{e}U(x) * \partial_\mu U^\dagger(x), \\
{}^hA'_\mu(x) &= U^\dagger(x) * A'_\mu(x) * U(x) + \frac{2i}{e}U^\dagger(x) * \partial_\mu U(x),
\end{aligned} \tag{3·18}$$

where we have defined

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}.$$

In the commutative limit we have $A^0_\mu(x) \to 0$ and $C_\mu(x) \to B_\mu(x)$ so that $Z'_\mu(x) \to -Z_\mu(x)$ and $A'_\mu(x) \to -A_\mu(x)$, the same spectrum as in the neutral gauge bosons sector of the GWS theory.

We write the gauge interactions of the chiral fermions as follows:

$$\begin{aligned}
&\bar\psi(x) * A(x) * \psi(x) - \bar\psi(x) * \psi(x) * B(x) \\
&\qquad = \frac{e}{2}\bar\nu(x) * \gamma^\mu(A_\mu(x) * \nu(x) - \nu(x) * A_\mu(x)) \\
&\qquad + \frac{e}{2}\bar e(x) * \gamma^\mu(A'_\mu(x) * e(x) - e(x) * A_\mu(x)) \\
&\qquad + Z_\mu\text{-interactions} + Z'_\mu\text{-interactions} + W_\mu\text{-interactions}.
\end{aligned} \tag{3·19}$$

Looking at Z_μ-interactions for the neutrino

$$\begin{aligned}
&\frac{g}{\cos\theta_W}[\frac{1}{2}(1 - \sin^2\theta_W)\bar\nu_L(x) * \gamma^\mu Z_\mu(x) * \nu_L(x) - \frac{1}{2}\sin^2\theta_W\bar\nu_R(x) * \gamma^\mu Z_\mu(x) * \nu_R(x) \\
&\qquad + \frac{1}{2}\sin^2\theta_W(\bar\nu_L(x) * \gamma^\mu\nu_L(x) + \bar\nu_R(x) * \gamma^\mu\nu_R(x)) * Z_\mu(x)],
\end{aligned} \tag{3·20}$$

where the Weinberg angle is defined by $\tan\theta_W = g'/g$, we conclude that ν_R interacts with Z_μ on noncommutative space-times, although it escapes the interaction in the commutative limit as it is gauge-singlet in GWS theory. In the commutative limit Eq. (3·14) is reduced to $\begin{cases} \nu(x) \to \nu(x), \\ e(x) \to U^{\dagger 2}(x)e(x), \end{cases}$ so that there is only one photon field $A_\mu = -A'_\mu$ and the leptons (ν, e) have the electric charges $(0, -e)$. On noncommutative space-times the unbroken symmetry is described by the gauge transformation (3·14). Consequently, in our noncommutative GWS model in the leptonic sector there are two 'photon' fields, A_μ, A'_μ, and two neutral massive gauge fields, Z_μ, Z'_μ. It can be seen from E. (3·19) that, in the tree level, only one 'photon', A_μ, couples to the neutrino, while both 'photons' interact with the electron. Similarly, the neutrino couples to Z_μ only but the electron does to both Z_μ and Z'_μ in the tree level. The neutral gauge fields become degenerate into the photon and Z^0, respectively, in the commutative limit. The structure of W_μ-interactions remain intact.

§4. Discussions

We have defined Connes' YM on noncommutative space-times. It contains more physical degrees of freedom than those in the commutative Connes' YM. We have considered a noncommutative GWS model in the leptonic sector. The model predicts that, in addition to the extra massive Higgses, there are two independent massless as well as two independent massive neutral gauge fields on noncommutative space-times. They become degenerate into the photon and Z^0, respectively, in the commutative limit.

In order to include color into the present scheme we may write

$$l(x) = \begin{pmatrix} l_L(x) \\ l_R(x) \end{pmatrix} \to g(x) * l(x) * U^\dagger(x), \quad g = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}, \quad g_R = \begin{pmatrix} U & 0 \\ 0 & U^\dagger \end{pmatrix},$$

$$q(x) = \left(\begin{pmatrix} q_L^r(x) \\ q_R^r(x) \end{pmatrix}, \begin{pmatrix} q_L^b(x) \\ q_R^b(x) \end{pmatrix}, \begin{pmatrix} q_L^g(x) \\ q_R^g(x) \end{pmatrix} \right) \to g(x) * q(x) * v^T(x),$$

where $v(x) \in C^\infty(M_4) \otimes M_3(\boldsymbol{C})$ with $v(x) * v^\dagger(x) = v^\dagger(x) * v(x) = 1$. The new gauge fields associated with $v(x)$ are the gluons. There is a ninth gluon $G^0_\mu(x)$ which is related to $A_\mu(x)$ via $G^0_\mu(x) = -(1/3)A_\mu(x)$ in the commutative limit in order to reproduce the correct assignment of the electric charges of quarks. This relation is to be imposed by hand as opposed to the limit $A'_\mu(x) \to -A_\mu(x)$ which is automatic in the leptonic sector. This may raise a problem in extending our noncommutative GWS model to a noncommutative standard model. This point

will be a subject in a forthcoming paper.

Non-commutativity of the operator or Moyal products implies that a noncommutative generalization of the conventional field theory model is not unique. As an example we consider a noncommutative QED for leptons (ν, e) with only a single Abelian gauge field A_μ. The relevant gauge transformation is given by

$$\begin{cases} \nu(x) \to U(x) * \nu(x) * U^\dagger(x), \\ e(x) \to U(x) * e(x). \end{cases}$$

The gauge couplings are determined as

$$\begin{cases} \bar{\nu}(x) * i\gamma^\mu (A_\mu(x) * \nu(x) - \nu(x) * A_\mu(x)), \\ \bar{e}(x) * i\gamma^\mu A_\mu(x) * e(x), \end{cases}$$

where the gauge field is assumed to transform like

$$A_\mu(x) \to U(x) * A_\mu(x) * U^\dagger + U(x) * \partial_\mu U^\dagger(x).$$

This is inconsistent, however, with the assumption that $\begin{pmatrix} \nu \\ e \end{pmatrix}_L$ is a doublet on noncommutative space-times. In this case both ν and e should receive the (unbroken) gauge transformation from both sides, since the neutrino is neutral. Our gauge transformation (3·14) is chosen to meet this assumption. But in that case we necessarily have two 'photons' which become a single photon in the commutative limit. There is a change in the spectrum of our noncommutative generalization of QED for the leptons (ν, e).

The non-commutativity parameter is very small so that we may work in the first-order approximation. We rewrite the ν-A_μ coupling in Eq. (3·19) to the first order in the non-commutativity parameter as

$$-i\frac{e}{2}\theta^{\rho\sigma}\partial_\rho\bar{\nu}(x)\gamma^\mu\partial_\sigma\nu(x)A_\mu(x),$$

where we have made the partial integration and used the antisymmetry of $\theta^{\rho\sigma}$. Similarly, the ν_R-interaction in Eq. (3·20) is approximated by

$$i\frac{g\sin^2\theta_W}{2\cos\theta_W}\theta^{\rho\sigma}\partial_\rho\bar{\nu}_R(x)\gamma^\mu\partial_\sigma\nu_R(x)Z_\mu(x).$$

Next consider the electron-interaction with two 'photons'. We can convert it to the familiar-looking one $-e\bar{e}(x)\gamma^\mu e(x)A_\mu(x)$ plus an additional one in the same approximation

$$\frac{e}{2}\theta^{\rho\sigma}\bar{e}(x)\gamma^\mu e(x)A_{\rho\mu\sigma}(x) \equiv \frac{e}{2}\bar{e}(x)\gamma^\mu e(x)\tilde{A}_\mu(x),$$

where we put ${A'}_\mu(x) = -A_\mu(x) + \theta^{\rho\sigma} A_{\rho\mu\sigma}(x)$. Although it is impossible to cast this extra one into the form $j^\mu(x)A_\mu(x)$, we can define the field strength for $\tilde{A}_\mu(x) = \theta^{\rho\sigma} A_{\rho\mu\sigma}(x)$ by $\tilde{F}_{\mu\nu}(x) = \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + e\theta^{\rho\sigma}\partial_\rho A_\mu(x)\partial_\sigma A_\nu(x)$ such that ${F'}_{\mu\nu}(x) = -F_{\mu\nu}(x) + \tilde{F}_{\mu\nu}(x)$ to the first order in $\theta^{\rho\sigma}$ *), where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + (e/2)\theta^{\rho\sigma}\partial_\rho A_\mu(x)\partial_\sigma A_\nu(x)$.

Or, it may be illegitimate to attempt to expand a noncommutative GWS model with respect to the non-commutativity parameter although the commutative limit can be discussed already in the Lagrangian level. We have not yet succeeded in finding an appropriate language of describing the change of the spectrum in our theory.

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Appendix A

In this Appendix we prove the trace formula $\mathrm{tr}\hat{T}(k) = [(2\pi)^2/\sqrt{\det\theta}]\delta^4(k)$. The 2-dimensional case was treated in Ref. 17).

We can always convert the (invertible) matrix $\theta = (\theta^{\mu\nu})$ to the canonical form [23)]

$$\theta = \begin{pmatrix} 0 & \theta_1 & 0 & 0 \\ -\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 \\ 0 & 0 & -\theta_2 & 0 \end{pmatrix}, \quad \theta_1\theta_2 \neq 0.$$

In this canonical form we have the following commutation relations

$$[\hat{x}^0, \hat{x}^1] = i\theta_1, \quad [\hat{x}^2, \hat{x}^3] = i\theta_2, \quad \text{others} = 0.$$

Using the annihilation and creation operators $\hat{\alpha} = (1/\sqrt{2\theta_1})(\hat{x}^0 + i\hat{x}^1), \hat{\alpha}^\dagger = (1/\sqrt{2\theta_1})(\hat{x}^0 - i\hat{x}^1), \hat{\beta} = (1/\sqrt{2\theta_2})(\hat{x}^2 + i\hat{x}^3)$ and $\hat{\beta}^\dagger = (1/\sqrt{2\theta_2})(\hat{x}^2 - i\hat{x}^3)$, which satisfy $[\hat{\alpha}, \hat{\alpha}^\dagger] = [\hat{\beta}, \hat{\beta}^\dagger] = 1$

*) To determine the propagator of $\tilde{A}_\mu$ we should retain a term quadratic in $\tilde{F}_{\mu\nu}$, which is higher order. The decomposition ${A'}_\mu(x) = -A_\mu(x) + \tilde{A}_\mu(x)$ defines $\tilde{A}_\mu(x)$ such that, in the first-order approximation, the infinitesimal gauge transformation is $\delta A_\mu = +(2/e)\partial_\mu\alpha - \theta^{\rho\sigma}\partial_\rho\alpha\partial_\sigma A_\mu$ and $\delta\tilde{A}_\mu = -2\theta^{\rho\sigma}\partial_\rho\alpha\partial_\sigma A_\mu$, where $U = (e^{i\alpha})_* = 1 + i\alpha$. Consequently, the sum $-e\bar{e}(x)\gamma^\mu e(x)A_\mu(x) + (e/2)\bar{e}(x)\gamma^\mu e(x)\tilde{A}_\mu(x)$ upon integration is gauge-invariant in the same approximation.

and $[\hat{\alpha}, \hat{\beta}] = [\hat{\alpha}, \hat{\beta}^\dagger] = 0$, we can write $\hat{x}^0 = \sqrt{\theta_1/2}(\hat{\alpha} + \hat{\alpha}^\dagger), \hat{x}^1 = (1/i)\sqrt{\theta_1/2}(\hat{\alpha} - \hat{\alpha}^\dagger), \hat{x}^2 = \sqrt{\theta_2/2}(\hat{\beta} + \hat{\beta}^\dagger)$ and $\hat{x}^3 = (1/i)\sqrt{\theta_2/2}(\hat{\beta} - \hat{\beta}^\dagger)$ so that we have $e^{ik_\mu \hat{x}^\mu} = e^{A+B+C+D}$, where $A = \sqrt{\theta_1/2}(ik_0 + k_1)\hat{\alpha}, B = \sqrt{\theta_1/2}(ik_0 - k_1)\hat{\alpha}^\dagger, C = \sqrt{\theta_2/2}(ik_2 + k_3)\hat{\beta}$ and $D = \sqrt{\theta_2/2}(ik_2 - k_3)\hat{\beta}^\dagger$. We next resort to the well-known formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \quad [A,B]: \ c\text{−number}$$

to obtain

$$\hat{T}(k) = e^{A+B+C+D} = e^A e^B e^C e^D e^{(\theta_1/4)(k_0^2+k_1^2)+(\theta_2/4)(k_2^2+k_3^2)}. \qquad (\mathrm{A} \cdot 1)$$

Since the trace is independent of the basis in the Hilbert space spanned by $\hat{x}^\mu$, we evaluate it in the coherent states basis

$$\mathrm{tr}\hat{T}(k) = (\frac{i}{2\pi})^2 \int dz dz^* d\zeta d\zeta^* \langle z, \zeta | \hat{T}(k) | z, \zeta \rangle e^{-|z|^2 - |\zeta|^2}, \qquad (\mathrm{A} \cdot 2)$$

where $|z, \zeta\rangle = e^{z\hat{\alpha}^\dagger} e^{\zeta \hat{\beta}^\dagger}|0\rangle$ with $\hat{\alpha}|0\rangle = \hat{\beta}|0\rangle = 0$. Substituting Eq. (A· 1) into Eq. (A· 2) we find

$$\mathrm{tr}\hat{T}(k) = (\frac{i}{2\pi})^2 \int dz dz^* d\zeta d\zeta^* e^X,$$

where $X = z^*\sqrt{\theta_1/2}(ik_0 - k_1) + z\sqrt{\theta_1/2}(ik_0 + k_1) + \zeta^*\sqrt{\theta_2/2}(ik_2 - k_3) + \zeta\sqrt{\theta_2/2}(ik_2 + k_3) - (\theta_1/4)(k_0^2 + k_1^2) - (\theta_2/4)(k_2^2 + k_3^2)$. Changing the variables by $z = x^0 + ix^1, z^* = x^0 - ix^1, \zeta = x^2 + ix^3, \zeta^* = x^2 - ix^3$ with $dz dz^* d\zeta d\zeta^* = (-2i)^2 d^4x$, we arrive at

$$\begin{aligned}
\mathrm{tr}\hat{T}(k) &= \frac{1}{\pi^2} \int d^4x e^{i[\sqrt{2\theta_1}(x^0 k_0 + x^1 k_1) + \sqrt{2\theta_2}(x^2 k_2 + x^3 k_3)]} e^{-(\theta_1/4)(k_0^2+k_1^2)-(\theta_2/4)(k_2^2+k_3^2)} \\
&= \frac{(2\pi)^4}{\pi^2} \delta(\sqrt{2\theta_1}k_0)\delta(\sqrt{2\theta_1}k_1)\delta(\sqrt{2\theta_2}k_2)\delta(\sqrt{2\theta_2}k_3) \\
&= \frac{(2\pi)^2}{\theta_1\theta_2}\delta^4(k) = \frac{(2\pi)^2}{\sqrt{\det\theta}}\delta^4(k).
\end{aligned}$$

Appendix B

Needless to say the gauge fields $A_\mu^{L,R}(x)$ of Eq. (3·15) must become traceless [18] in the commutative limit. The purpose of this Appendix is to prove this statement in our formulation.

By writing the elements of $C^\infty(M_4) \otimes \boldsymbol{H}$ in Eq. (3·4) as $a_i^L(x) = \begin{pmatrix} \alpha_i(x) & \beta_i(x) \\ -\beta_i^*(x) & \alpha_i^*(x) \end{pmatrix}$ and

$b_i^L(x) = \begin{pmatrix} \gamma_i(x) & \delta_i(x) \\ -\delta_i^*(x) & \gamma_i^*(x) \end{pmatrix}$, we have $(A_\mu^L)_{11}(x) = \sum_i(\alpha_i(x) * \partial_\mu \gamma_i(x) - \beta_i(x) * \partial_\mu \delta_i^*(x))$, while $(A_\mu^L)_{22}(x) = \sum_i(-\beta_i^*(x) * \partial_\mu \delta_i(x) + \alpha_i^*(x) * \partial_\mu \gamma_i^*(x))$. Because of the $*$-product they are independent. However, in the commutative limit, we can omit the $*$-symbol so that $(A_\mu^L)_{11}(x) = \sum_i(\alpha_i(x)\partial_\mu\gamma_i(x) - \beta_i(x)\partial_\mu\delta_i^*(x)) = -\sum_i(\alpha_i^*(x)\partial_\mu\gamma_i^*(x) - \beta_i^*(x)\partial_\mu\delta_i(x)) = -(A_\mu^L)_{22}(x)$, where we have used the anti-hermiticity.

On the other hand, by our choice of $g_R(x)$ $(A_\mu^R)_{11}(x)$ and $B_\mu(x)$ enjoy the same gauge transformation law so that we should put $c_i^*(x) = (a_i^R)_{11}(x)$ and $d_i^*(x) = (b_i^R)_{11}(x)$, yielding the equality $B_\mu(x) = \sum_i c_i^*(x) * \partial_\mu d_i^*(x) = \sum_i(a_i^R)_{11}(x) * \partial_\mu(b_i^R)_{11}(x) = (A_\mu^R)_{11}(x)$. In contrast, $(A_\mu^R)_{22}(x) = \sum_i(a_i^R)_{22}(x) * \partial_\mu(b_i^R)_{22}(x)$ is not related with $B_\mu(x)$ since $(a_i^R)_{22}(x) = (a_i^{R*})_{11}(x)$ and $(b_i^R)_{22}(x) = (b_i^{R*})_{11}(x)$, and $\sum_i(a_i^{R*})_{11}(x) * \partial_\mu(b_i^{R*})_{11}(x)$ is not equal to $-\sum_i(a_i^R)_{11}(x) * \partial_\mu(b_i^R)_{11}(x) = -(A_\mu^R)_{11}(x)$. As in the previous case, however, in the commutative limit we can omit the $*$ symbol and $\sum_i(a_i^{R*})_{11}(x)\partial_\mu(b_i^{R*})_{11}(x) = -\sum_i(a_i^R)_{11}(x)\partial_\mu(b_i^R)_{11}(x)$ by anti-hermiticity. Hence, $(A_\mu^R)_{11}(x) = -(A_\mu^R)_{22}(x)$ in the commutative limit.

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